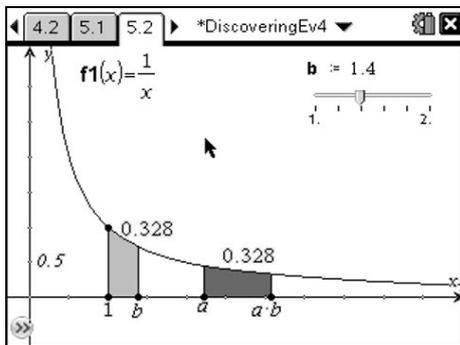
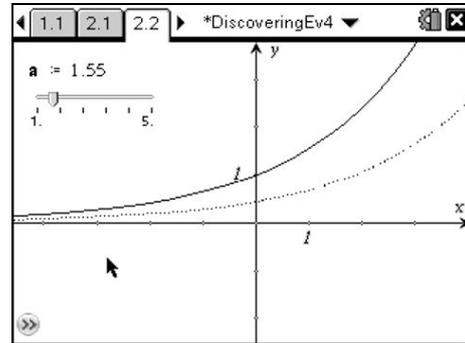


Discovering e

<p>Mathematical Content: Euler's number Differentiation of exponential functions Integral of 1/x Sequences & series</p>	<p>Technical TI-Nspire Skills: Manipulation of geometric constructions Use of sliders Questions & answers Lists</p>
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This activity introduces Euler's number e from a variety of perspectives.

The first approach looks at how e can be defined as the base of the special exponential function that is its own derivative.



The second section explores the integration of the $1/x$ function. It builds through the process of observing that the area under the curve behaves like a logarithmic function, justifying this statement and then it moves towards the observation that the base of this logarithmic function turns out to be e .

Finally there are two extension tasks, one looking at the definition of a series whose limit is e , and another looking at a sequence whose limit is e .

As a whole, this activity should give your students insight into the mathematical origins of this interesting and pervasive number.

Series:

$$\text{Firstly } e = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n!} \right) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Use the calculator on the next page to find the value of e for different values of the upper bound

Limits of Sequences:

The number e can also be defined as the limit of the sequence:

$$u_n = \left(1 + \frac{1}{n} \right)^n$$

Look at the spreadsheet on the next page and explore the rate at which the sequence converges.

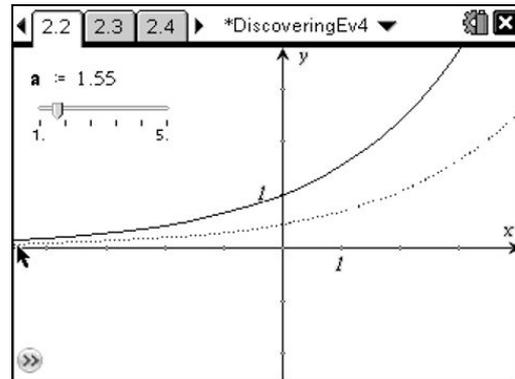
Discovering e – Student worksheet

In this activity you will explore the origins of Euler's Number sometimes referred to simply as e .

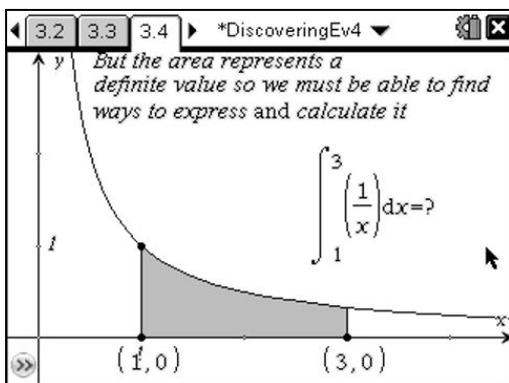
Task 1:

To begin the activity you will need to open the DiscoveringE.tns file on your handheld. The opening pages introduce the task and explain what to do on page 2.2. To move to the next page, press (ctrl)▶.

On page 2.2 you are presented with graphs of an exponential function with base a and its derivative. Manipulate the slider and identify 3 different cases where the derivative is less than, greater than and equal to the function. For what value of a is the function equal to the derivative for all values of x ?



By the end of this task on page 2.6 you will have found one definition for Euler's Number.



Task 2:

In this second, longer, task you will meet an alternative definition for Euler's Number this time coming from the area of integration. Starting on page 3.1, you will explore the definite integral of $\frac{1}{x}$.

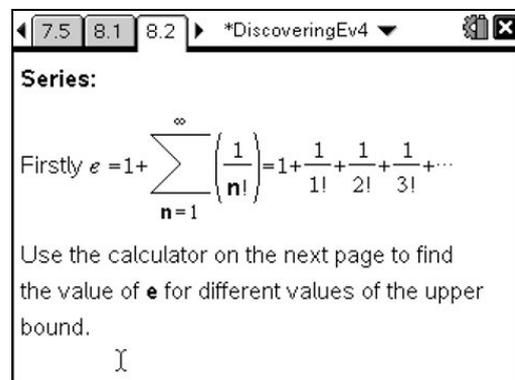
Follow the instructions on pages 3.1 to 7.5 to explore the various properties of this interesting integral before finally exploring how it is related to Euler's Number.

Extension Tasks:

In this final section you will explore how Euler's Number can be defined as the limit of a series and sequence, which seem completely unrelated to the other places we have discovered Euler's Number so far. On page 8.3 when you are looking at the Series, you can force the handheld to give you decimal (rather than fractional) answer by pressing (ctrl)(enter), and can reuse the previous entry by pressing ▲▲(enter), which will copy then entry down allowing you to edit it and use a different upper limit

On page 8.6 if you need to move between the upper and lower panes you can do this by pressing (ctrl)(tab).

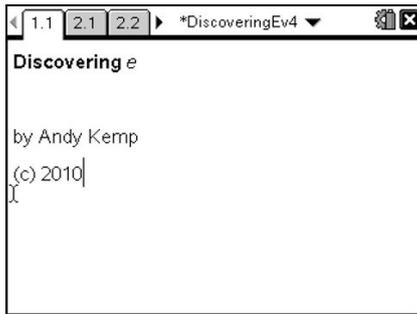
Working through this activity should hopefully have given you some insight into the mathematical richness hidden in the seemingly innocent looking number e .



Discovering e – Detailed notes

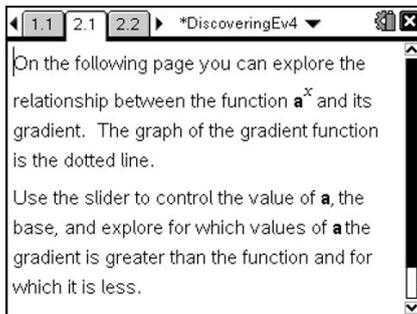
These notes briefly describe the content of each page and draw attention to any important elements

Page 1.1



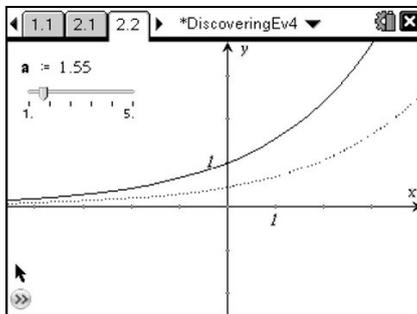
This page is simply a title page

Page 2.1



This page introduces the first task of exploring the relationship between a^x and its gradient function

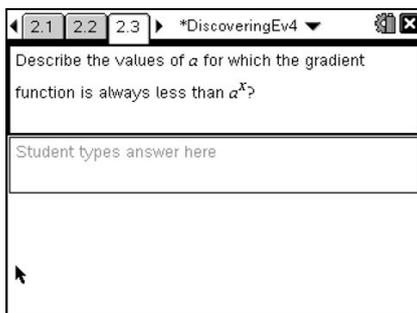
Page 2.2



On this page the student can manipulate the base of the exponential function by adjusting the slide.

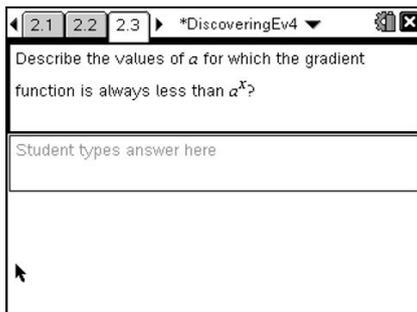
The gradient function is the dotted graph.

Page 2.3



The student should observe that the gradient function is always less than the function when a is less than 2.7

Page 2.4



The student should observe that the gradient function is always greater than the function when a is greater than 2.7

Page 2.5

It appears from examining the graph that there is a value of **a** for which the gradient of a^x is a^x .

This value of **a** is called Euler's number and is often written as *e*.

From the graphs you can see that:

$e \approx 2.7$

This page explains what has been discovered on the previous three pages.

Page 2.6

This leads to the surprising result that:

$$\frac{d}{dx} e^x = e^x$$

This is one definition for the value of *e*

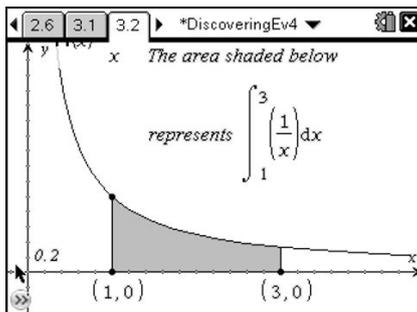
This page summarises the first definition for Euler's Number.

Page 3.1

In this next section you will explore the integral of $\frac{1}{x}$. Using standard techniques for integration it is not possible to find the value $\int_1^3 \left(\frac{1}{x}\right) dx$. However, by looking at the next page you can clearly see it has a fixed value.

This page introduces the second task, which explores the integral of $\frac{1}{x}$.

Page 3.2



This page shows the definite integral between 1 and 3 of $\frac{1}{x}$. It is clearly a finite area.

Page 3.3

Explain why we cannot use the typical rule that: $\int (kx^n) dx = \frac{kx^{n+1}}{n+1}$ to integrate $\int \left(\frac{1}{x}\right) dx$

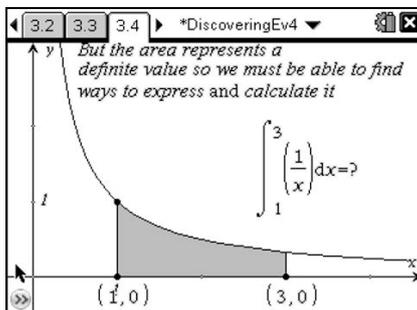
Student types answer here

The student should hopefully observe that using the standard polynomial integral rule would lead to:

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^0}{0} = \frac{1}{0}$$

But division by zero is undefined...

Page 3.4



This page reminds the student that despite the standard rule leading to an undefined value the integral must have a defined value: we could measure it using a numerical method like the trapezium rule.

Page 3.5

It can be seen by looking at the graph that

$$\int_1^a \left(\frac{1}{x}\right) dx \text{ takes a definite value for all } a > 1.$$

So it must be possible to find a way to express this area as a function of **a**.

In fact the integral must have a value for any upper limit (greater than 1) as the reciprocal function is well behaved as x moves away towards infinity. So the area must remain finite for any upper finite limit.

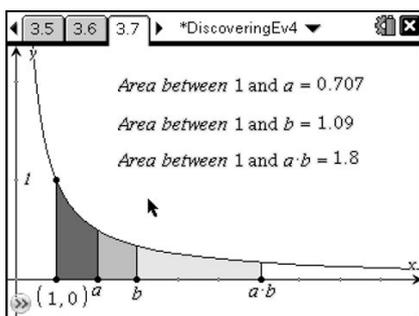
Page 3.6

The area under the $\frac{1}{x}$ graph has a very interesting property.

Use the graph on the next page and move the points **a** and **b** to see if you can spot what is going on...

This page introduces the idea that this integral has an interesting property that may help express the area of the integral.

Page 3.7



Students should hopefully notice that, however they move the points **a** and **b**, the areas between 1 and **a** and 1 and **b** add together to give the area between 1 and **ab**.

Page 3.8

Can you describe the relationship you found on the previous page?

Student types answer here

Students should conclude something like:

$$\int_1^a \frac{1}{x} + \int_1^b \frac{1}{x} = \int_1^{a \cdot b} \frac{1}{x}$$

Page 3.9

So, to simplify the exploration, use $I(a)$ to refer to the integral $\int_1^a \left(\frac{1}{x}\right) dx$.

So $I(x)$ represents the area between 1 and x.
(Note: it is always between 1 and x.)

At this point we simplify the notation by making:

$$\int_1^x \frac{1}{t} dt = I(x)$$

Page 3.10

Using the new notation, the relationship can be expressed as:

$$I(a) + I(b) = I(a \cdot b)$$

Why should it be the integral of **a times b**?

Over the next few pages this observation is justified.

This change of notation allows us to rewrite the property expressed on page 3.8 in a slightly different form, which students may recognise as being similar to one of the properties of logarithms.

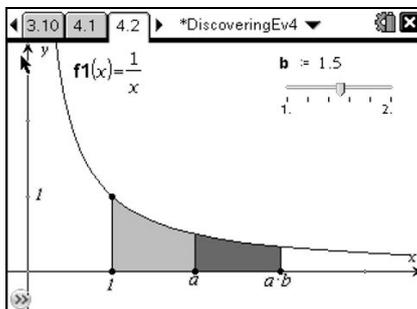
Page 4.1

Start by considering the graph on the following page and convince yourself that:

$$I(a) + \int_a^{ab} \left(\frac{1}{x}\right) dx = I(ab)$$

Now we begin the process of justifying the observation made on Page 3.8 and refined on page 3.10, by considering the following property,

Page 4.2



It is clear from the graph that the integral between 1 and **a** plus the integral between **a** and **ab** is equivalent to the integral between 1 and **ab**.

Page 5.1

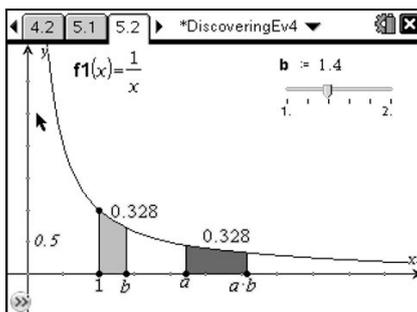
Next consider why:

$$\int_a^{ab} \left(\frac{1}{x}\right) dx = \int_1^b \left(\frac{1}{x}\right) dx = I(b)$$

Why is this a surprising result?
Explore it by dragging the point **a**, and the slider for **b** on the next page.

Next we will explore the following more surprising property that the integral between **a** and **ab** is the same as the integral between 1 and **b**

Page 5.2



Graphically we can see that by dragging the point **a** about or by changing the value of **b** the two areas remain the same size.

Page 5.3

You may want to consider how you could prove algebraically that:

$$\int_a^{ab} \left(\frac{1}{x}\right) dx = \int_1^b \left(\frac{1}{x}\right) dx = I(b)$$

Hint: consider the effect of the substitution $x=az$, but this is left to you to explore.

The proof of this statement is left for the student to explore independently but is included here for completeness:

If we let $x = az$ then $\frac{dx}{dz} = a$ so,

$$\int_a^{ab} \frac{1}{x} dx = \int_{x=a}^{x=ab} \frac{1}{az} \frac{dx}{dz} dz = \int_{z=1}^b \frac{a}{az} dz = \int_1^b \frac{1}{z} dz =$$

Page 5.4

Pulling together the results from pages 4.1 and 5.1:

$$I(a) + \int_a^{ab} \left(\frac{1}{x}\right) dx = I(ab) \text{ and } \int_a^{ab} \left(\frac{1}{x}\right) dx = I(b)$$

So $I(a) + I(b) = I(ab)$

This page pulls together the two properties to justify the result:

$$I(a) + I(b) = I(ab)$$

Page
6.1

You should recognise this as being similar to one of the laws of logs:

$$\int \log_{\alpha}(a) + \log_{\alpha}(b) = \log_{\alpha}(a \cdot b)$$

Where α is called the base of the logarithm.

Spend a few minutes convincing yourself that the function $I(x)$ satisfies the rest of the properties of a log function as outlined on the next page.

This page draws the student's attention to the fact that the property they have just justified is the equivalent to one of the law of logarithms,

Page
6.2

Laws of Logs:

- 1) $I(1) = 0$
- 2) $\int I(a) + I(b) = I(a \cdot b)$
- 3) $\int I(a) - I(b) = I\left(\frac{a}{b}\right)$
- 4) $\int I(a^n) = n \cdot I(a)$

Note: You have already proved Law 2.

This page outlines the other laws of logs and asks the students to justify them, proofs of these statements are included here for completeness.

- 1) $I(1) = \int_1^1 \frac{1}{x} dx = 0$ as the upper and lower limit are the same
- 2) Already completed
- 3) Consider $I\left(\frac{a}{b}\right) + I(b)$

Using rule 2 on this gives:

$$I\left(\frac{a}{b}\right) + I(b) = I\left(\frac{a}{b} * b\right) = I(a), \text{ so}$$
$$I(a) - I(b) = I\left(\frac{a}{b}\right) \text{ as required}$$

- 4) This statement is easy to prove when n is an integer as:
 $I(a^n) = I(a \cdot a \cdots a \cdot a) = I(a) + \cdots + I(a) =$
We can show this for any rational number by letting $n = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $a^n = u$

so:

$$q I(u) = I(u^q) = I\left(a^{\frac{p}{q} \cdot q}\right) = I(a^p) = p I(a)$$

$$\text{So } I\left(a^{\frac{p}{q}}\right) = \frac{p}{q} I(a) \text{ as required.}$$

Page
7.1

Now that we have established that the area under the $\frac{1}{x}$ function is described by a logarithmic function we finally need to establish the base of the log function.

Log functions are defined so that:

$$\log_a(a) = 1 \text{ where } a > 1$$

This page reminds students that the base of any logarithm is such that $\log_a a = 1$

Page
7.2

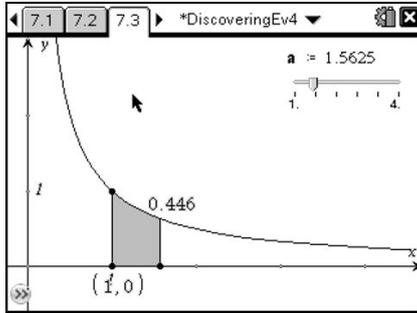
So we must find a value of a for which

$$I(a) = \int_1^a \left(\frac{1}{x}\right) dx = 1$$

Use the slider on the next page to explore the value of a for which this is true.

So relating this fact back to the integral leads back to the graphical environment where we must find the value of a such that $I(a)=1$

Page 7.3



On this page students should adjust the value of a to find a value for which the area is 1.

Students should find that this occurs when a is about 2.7

Page 7.4

Do you recognise the value of a ?

That's right, it is same number e we found earlier so we have another definition for e :

$$\int_1^e \left(\frac{1}{x}\right) dx = 1$$

Students should hopefully remember that this value of 2.7 is the same as the value they found in the first task.

Page 7.5

Also the function $I(x)$ is called 'log to the base e ' or the 'natural logarithm' and is usually denoted by $\ln(x)$.

So we can conclude that the integral of $1/x$ (between appropriate limits) can be expressed by a log function with a base of e .

Page 8.1

Extension:

Surprisingly e appears in all sorts of areas of mathematics. Two other definitions come from the areas of sequences and series...

$$I$$

This page introduces the extension section.

Page 8.2

Series:

Firstly $e = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n!}\right) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

Use the calculator on the next page to find the value of e for different values of the upper bound.

Another way of finding the value of e is by considering the limit of the series.

Page 8.3

$1 + \sum_{n=1}^I \left(\frac{1}{n!}\right)$

0/99

On this page students are encouraged to try different upper limits to this series to see how quickly it converges to the value of e

You can force the handheld to give you decimal (rather than fractional) answer by pressing **(ctrl) (enter)**, and can reuse the previous entry by pressing **▲ ▲ (enter)**, which will copy the previous entry allowing you to edit it and use a different upper limit.

Page 8.4

Limits of Sequences:
The number e can also be defined as the limit of the sequence:

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

Look at the spreadsheet on the next page and explore the rate at which the sequence converges. γ

This page suggests an alternative definition for e this time as the limit of a sequence as n tends to infinity

Page 8.5

n		
1	1.	2.
2	2.	2.25
3	3.	2.37037
4	4.	2.44141
5	5.	2.48832

Formula bar: $= (1 + 1/n)$

On this page students can explore the rate of convergence by scrolling through the values.

Students should notice that this sequence takes a long time to get close to the value of e (in fact it takes over 70 terms before we even get a number starting 2.7...)

Page 8.6

Now use the calculator window below to explore the value for large values of n :

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$

$f(1000)$

Done

1/99

On the next page we have defined a function

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$

Students can try different values of n to look at what happens to the sequence as n gets very large...

Page 8.7

Hopefully, these various examples have helped you to appreciate the mathematical richness that is stored in this seemingly simple number e ..

This final page summarises the activity.